Revisiting the choice problem

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Abstract

To do...

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1 Introduction

The goal of this study is to build a best choice recommendation from a set of alternatives X. The preference relation on the set X is constructed as a pairwise comparison indicating to which degree an alternative is at least as good as another. Such an outranking relation R is generally neither complete [2], antisymetrical [3] nor transitive [4].

Roy (Roy85) defines the $P.\alpha$ choice problem as a help to determine the "best" alternative. In general, the search for this alternative can in a first step be resumed to the ellicitation of a subset of alternatives which is as restricted as possible. This set Y is meant to help the decision maker to get closer to the selection of the best alternative.

In the case where the outranking relation is a weak order, the choice recommendation is trivially given by the maximum alternative or the equivalence

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[2] $x \,\mathrm{R}\, y$ or $y \,\mathrm{R}\, x$
[3] $x \,\mathrm{R}\, y$ and $y \,\mathrm{R}\, x \Rightarrow x = y$
[4] $x \,\mathrm{R}\, y$ and $y \,\mathrm{R}\, z \Rightarrow x \,\mathrm{R}\, z$

class of all maximal alternatives. If the outranking relation is a partial order, all the alternatives of the maximal equivalence classes are natural candidates for the best choice recommendation.

If the outranking relation is not transitive, things become less trivial.

`...compléter encore un peu l'intro par les noyaux et les hypernoyaux, et puis par la structure générale du papier`

2 On the $P.\alpha$ problem

This Section presents the "best" choice problem. We first focus on the optimisation problem, which is a particular case.

2.1 The optimisation problem

In the beginning of the developpment of operational research, the aim was to determine an optimal decision (the optimum) by basing it on models which describe an objective reality (Roy00). The concept of the search for an optimum has entered the everyday language. People often talk about the optimum or about optimising something. An optimal decision is a decision for which every other possible decisions is strictly worse or at most equivalent. The search for an optimum is therefore based on the following implicit existence axiom (Roy81):

> In any situation which necessarily involves a decision, there exists at least one decision which, with sufficient time and means, may be objectively proved as being optimal whilst remaining neutral in relation to the decision process.

In (Roy81), three fundamental conditions are elicitated to give sense to the concept of optimum and guarantee the existence of at least one optimal alternative. In particular, the set X of alternatives must be given beforehand and cannot be changed during the decision process. Furthermore the alternatives of X must all be comparable and the comparision relation must be transitive.

This last condition is very strong and may not be realistic in many cases. In particular, in case of a multiple criteria framework, if alternatives are evaluated by a pairwise comparision procedure, the resulting outranking relation is not necessarily transitive.

The $P.\alpha$ problem is different from the quest for the optimum. Roy (Roy85) defines the $P.\alpha$ choice procedure as an aid to choose the "best" alternative or to elaborate a selection procedure. In the following description, the optimisation problem emerges as a special case of the choice procedure.

The goal of the "best" choice problem $P.\alpha$ is to select a single "best" alternative. The investigation is therefore oriented towards the ellicitation of a subset Y of X which is as restricted as possible. In order to tend towards an optimum, a small set is required. This set Y is meant to enlighten the decision maker on the next important step of the process. It is important to note here that the set of alternatives X could be revisable or transitory. Furthermore, nothing is said about the transitivity of the relation which allows to compare the alternatives of X.

The objective of a choice procedure is not to obtain a single "best" alternative at any cost. As Roy explains, it is important that the not selected alternatives are rejected for well motivated reasons. Therefore, instead of forcing the procedure to elicitate a single best alternative, it is preferable to obtain a set of elements Y, as long as this choice can be strongly justified. This means that there must be enough arguments to reject the most possible alternatives from the final subset Y.

Starting from these observations, Roy defines the concept of choice in the $P.\alpha$ problem as a subset Y of X which has two main characteristics $\mathcal{R}_\mathbf{1}$ and $\mathcal{R}_\mathbf{2}$:

- $\mathcal{R}_\mathbf{1}$ Each alternative which in not selected must be considered as worse as at least one alternative of Y.
- $\mathcal{R}_\mathbf{2}$ The subset of retained alternatives Y must be as small as possible.

This means that the alternatives of Y are potential best decision candidates. A deeper analysis, restricted to Y may eventually reduce the numbre of decision candidates, or in the ideal case let appear the optimum. Finally, if $Y = X$ or Y is empty, no best choice recommendation can be provided.

From a methodological point of view, Roy (Roy85) uses the concept of *dominant kernel* of an outranking digraph to solve the choice problem. A dominant kernel is a dominant and independent set of alternatives. Nevertheless, the existence of a unique dominant kernel in a digraph is only verified if the graph has no circuits. Therefore, in order to avoid the emptyness or the multiplicity of the set of dominant kernels in a digraph, two approaches are suggested. The first one proposes to detect the circuits of maximal order in the outranking digraph and to reduce some or all of them to an equivalence class (ELECTRE I). The second approach tends to increases the discrimination between certain alternatives by removing certain outranking relations in order to break circuits (ELECTRE IS). Both approaches allow to obtain a unique dominant kernel, and consequently a unique best choice recommendation.

2.3 Flaws

Roy's procedure presents some disadvantages. First of all, in the ELECTRE-like procedures for the "best" choice recommendation, modifications are performed on the outranking digraph in case it has circuits. As a direct consequence the information describing the original problem is altered. A second flaw concerns the use of the kernel as a best choice. A kernel is necessarily a set of independent alternatives. This means that any two alternatives are incomparable (by an absence of a relational link between them). This is a very strong condition which can lead in some cases, when the graph contains circuits, to no solution to the $P.\alpha$ problem.

3 New foundations for the "best" choice problem

3.1 Difficulties using kernels

Nevertheless, in these classical approaches, a modification of the set of alternatives and / or the outranking relation has to be performed in order to obtain a choice recommendation. Another approach could be to modify the outranking relation in order to make it transitive. In that case, as we will see later in Section 5 in Property 6, there exists at least one dominating kernel.

The multiplicity of the kernels represents a further difficulty. One solution could be to take the union of the kernels as a best choice recommendation. Nevertheless, the union of kernels is not a kernel and Roy's second condition $\mathcal{R}_\mathbf{2}$ can no longer be verified.

Therefore, instead of asking the set Y to be as small as possible, we prefer to ask that the set of not selected alternatives by as large as possible and that a best choice recommendation should be minimal for the choice problem (i.e. not contain another best choice recommendation).

3.2 New principles

We can now write new principles for a best choice recommendation Y:

- $\mathcal{BMR}_\mathbf{1}$ Each alternative which is not selected must be considered as worse as at least one alternative of Y $(= \mathcal{R}_\mathbf{2})$.
- $\mathcal{BMR}_\mathbf{2}$ The set of rejected alternatives must be maximal.
- $\mathcal{BMR}_\mathbf{3}$ The best choice recommendation cannot contain another smaller best choice recommendation.

3.3 An extension to the kernel of a digraph

The following Section shows the properties of the kernels in digraphs and presents a generalised kernel, namely the hyper-kernel, which will be used later in the $P.\alpha$ problem.

4 A procedure for solving the "best" choice problem

4.1 On hyper-kernels

4.2 Solving the choice problem

4.3 Examples

5 On kernels and hyper-kernels in graphs

Let us first start by introducing a few important concepts necessary for our future discourse.

5.1 The outranking graph

Let X be a finite set of alternatives and R be a binary outranking relation defined on X. We suppose that R is characterised by an ordinal valued bipolar credibility domain $\widetilde{\mathrm{R}} : X \times X \to \mathcal{L} = \{-m, \ldots, 0, \ldots, m\}$. If x and y are two alternatives of X, $\widetilde{\mathrm{R}}(x, y) > 0$ means that $x \,\mathrm{R}\, y$ is more true than false; $\widetilde{\mathrm{R}}(x, y) < 0$ means that $x \,\mathrm{R}\, y$ is more false than true; $\widetilde{\mathrm{R}}(x, y) = 0$ means that $x \,\mathrm{R}\, y$ is either true or false. Let $\widetilde{G}(X, \widetilde{\mathrm{R}})$ be the digraph representing this $\mathcal{L}$-valued outranking relation. The binary outranking relation R is build from $\widetilde{\mathrm{R}}$ as follows:

$$\forall (x, y) \in X \times X : (x, y) \in \mathrm{R} \iff \widetilde{\mathrm{R}}(x, y) > 0$$

As a consequence, we call $G(X, \mathrm{R})$ the *strict median cut* crisp digraph associated to $\widetilde{G}$.

The *order* n of the digraph $\widetilde{G}(X, \widetilde{\mathrm{R}})$ is given by the cardinality of X, whereas the *size* m of $\widetilde{G}$ is given by the cardinality of $\widetilde{\mathrm{R}}$. As X is a finite set of n alternatives, the size m of the digraph $\widetilde{G}$ is also finite.

A digraph $\widetilde{G}$ is said *empty* if and only if $\widetilde{\mathrm{R}} = \emptyset$. On the opposite, a digraph $\widetilde{G}$ is said *complete* and noted K_n if and only if $\widetilde{\mathrm{R}} = X \times X$. A digraph $\widetilde{G}$ of order n is said to be *$\mathcal{L}$-connected* if the symmetric and transitive closure of its associated strict median cut crisp digraph G equals K_n.

In the sequel, we assume that the digraphs are $\mathcal{L}$-connected, i.e. that they don't have any isolated nodes.

We note C_k an intransitive oriented cycle of order k of a digraph. C_k is also called a *circuit* of order k. Let A be a subset of nodes of X. The graph $\widetilde{G}'(A, \widetilde{\mathrm{R}}|_A)$, where $\widetilde{\mathrm{R}}|_A$ is the relation $\widetilde{\mathrm{R}}$ restricted to the nodes of A, is called an *induced* subgraph of $\widetilde{G}$.

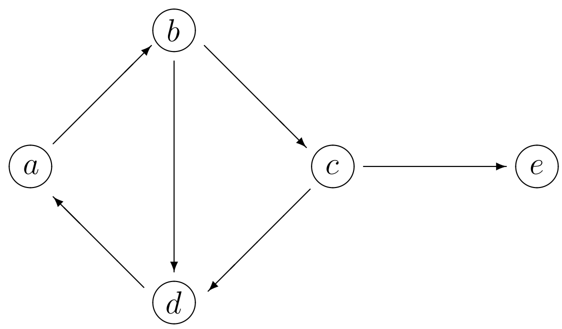

Fig. 1. Reference example

Reference example *Throughout this paper we will use a reference example of 6 alternatives which is given in the following table.*

$\widetilde{\mathrm{R}}$	*a*	*b*	*c*	*d*	*e*
a	*10*	*6*	*-10*	*-7*	*-9*
b	*-8*	*10*	*9*	*10*	*0*
c	*-10*	*-10*	*10*	*6*	*9*
d	*8*	*-8*	*-10*	*10*	*-7*
e	*-10*	*-9*	*-7*	*-8*	*10*

The strict median cut *crisp digraph $G(X, \mathrm{R})$ associated to the relation $\widetilde{\mathrm{R}}$ is given on figure 1. It is a graph of order* 5 *and size* 6*, where* $X = \{a, b, c, d, e\}$

and $\mathrm{R} = \{(a,b),(b,c),(b,d),(c,d),(c,e),(d,a)\}$*. The nodes* $\{a,b,d\}$ *induce a graph which is a circuit* C_3*. One should note here that it is impossible to represent the richness of the bipolar valued characterisation* $\widetilde{\mathrm{R}}$ *in the crisp graph* G*.*

5.2 Dominant and absorbent choices

A non empty subset Y of X is called a *choice* in $\widetilde{G}$. Such a choice Y is said to be *dominant* if and only if either, $Y = X$, or $x \notin Y \;\Rightarrow\; \exists y \in Y : \widetilde{\mathrm{R}}(y,x) \geq 0$. Similarly, a choice Y is said to be *absorbent* if and only if either $Y = X$, or $x \notin Y \;\Rightarrow\; \exists y \in Y : \widetilde{\mathrm{R}}(x,y) > 0$.

A choice, graph or relation is said to be *qualified* if it has a certain caracteristic (the qualification). For example the property of dominance is a particular qualification for a choice.

A qualified choice Y is called *minimal* if and only if $\forall Y' \subseteq Y$ (Y' and Y equally qualified), we have $Y \subseteq Y'$. A qualified choice Y is called *maximal* if and only if $\forall Y' \supseteq Y$ (Y' and Y equally qualified) we have $Y \supseteq Y'$.

Reference example *If we consider the dominant or absorbant caracteristics as qualifications, we can now consider minimal dominant and minimal absorbent choices. In our reference example, we note four minimal dominant choices:* $\{a,b,e\}$*,* $\{a,c\}$*,* $\{b,c,d\}$*, and* $\{b,d,e\}$ *and four minimal absorbent choices:* $\{a,b,e\}$*,* $\{a,c,e\}$*,* $\{a,d,e\}$*, and* $\{b,d,e\}$ *(see figure 1).*

The *dominated closed neighborhood* of a node $x \in X$, denoted $\Gamma^+(x)$, is the union of x itself and of all nodes $y \in X$ such that $\widetilde{\mathrm{R}}(x,y) > 0$. The *absorbed closed neighborhood* of node x is denoted similarly $\Gamma^-(x)$ and consists of the union of x and all nodes $y \in X$ such that $\widetilde{\mathrm{R}}(y,x) > 0$. The dominated (absorbed) neighborhood of a choice Y is denoted $\Gamma^+(Y)$ ($\Gamma^-(Y)$) and consists of the union of all dominated (absorbed) neighborhoods of the nodes of Y.

The dominated (absorbed) *closed private neighborhood* of a node x in a given choice Y is written $\Gamma^+_Y(x)$ ($\Gamma^-_Y(x)$) and consists of $\Gamma^+(x) \setminus \Gamma^+(Y \setminus \{x\})$ (resp. $\Gamma^-(x) \setminus \Gamma^-(Y \setminus \{x\})$).

Reference example *Node* b *in the choice* $\{a,b,e\}$ *has closed private neighborhoods* $\Gamma^+_{\{a,b,e\}}(b) = \{c,d\}$ *and* $\Gamma^-_{\{a,b,e\}}(b) = \{b\}$ *(see figure 1).*

Property 1 *Let* $\widetilde{G} = (X, \widetilde{\mathrm{R}})$ *be a digraph. In a minimal dominant (absorbent) choice* Y*, every node* $y \in Y$ *has a non empty dominated (absorbed) private neighborhood* $\Gamma^+_Y(y)$ *(*$\Gamma^-_Y(y)$*).*

Proof: Let us suppose that there exists indeed some node $y' \in Y$ such that $\Gamma^+_Y(y)$ ($\Gamma^-_Y(y)$) is empty. In this case we may remove this node from Y while $Y \setminus \{y'\}$ remains dominant (absorbent). This contradicts the hypothesis that Y is minimal with this qualification. □

Property 2 *Every finite digraph contains at least one minimal dominant (absorbent) choice.*

Proof: By definition, every digraph admits at least the trivial dominant and absorbent choice $Y = X$. If this choice is not minimal, we can progressively drop a finite number of nodes which have an empty private neighbourhood until we necessarily get a dominant (absorbent) choice where all nodes have a non empty private neighborhood. This choice is still dominant (absorbant) and is mimimal with this quality. □

Demonstration of property 2 is a constructive demonstration and it gives a first algorithm for computing all minimal dominant (absorbent) choices in a given digraph. We start from the greedy choice and try all possible droppings of "redundant" nodes until we reach a minimal dominant (absorbant) choice. An opposite strategy gives a second algorithm where one tries to construct a dominant (absorbent) choice starting from the smallest possible choices, the singletons. Here we try to add all possible "irredundant" nodes , i.e. nodes which have non empty private neighborhoods.

5.3 Irredundant and independent choices

A dominant (absorbent) choice $Y \subseteq X$ that contains only nodes which have a non empty dominated (absorbed) private neighborhood is said to be *irredundant*. All minimal dominant and absorbent choices are in fact irredundant, they are even maximal for this quality.

Property 3 *(CHM78) Let $\widetilde{G}(X, \widetilde{\mathrm{R}})$ be a digraph. A choice Y in $\widetilde{G}$ is minimal dominant (absorbent) if and only if it is dominant (absorbent) and irredundant.*

Property 4 *(BC79) Let $\widetilde{G}(X, \widetilde{\mathrm{R}})$ be a digraph. A choice Y in $\widetilde{G}$ is minimal dominant (absorbent) if and only if it is maximal irredundant.*

Proof: Let us suppose that Y is *minimal dominant* (*absorbent*) but not maximal irredundant. This implies that there exists a node $x \in X \setminus Y$ such that $Y \cup \{x\}$ is irredundant, i.e. $\Gamma^+(Y)$ (resp. $\Gamma^-(Y)$) is a proper subset of

$\Gamma^+(Y \cup \{x\})$ (resp. $\Gamma^-(Y \cup \{x\})$). This contradicts however the fact that Y is dominant (absorbent).

The other way round, let us suppose that Y is maximal *irredundant* but not minimal dominant (absorbent). This implies that there exists a node $y \in Y$ such that $Y \setminus \{y\}$ remains dominant (absorbent), i.e. this y cannot have a private neighborhood with respect to Y. This contradicts however the hypothesis that Y is irredundant. $\square$

A choice is called *independent* if and only if either, Y is a singleton, or $\forall x, y \in Y : \widetilde{\mathrm{R}}(x,y) < 0$. One should notice here that independence is not based on the negation of the positiveness of the outranking. Such a negation would also include the couples of alternatives (x, y) for which $\widetilde{\mathrm{R}}(x,y) = 0$ holds. Therefore, independence cannot be considered solely in the strict median cut digraph associated to $\widetilde{G}$. Consequently one can see that the median element 0 plays a very particular role in $\mathcal{L}$.

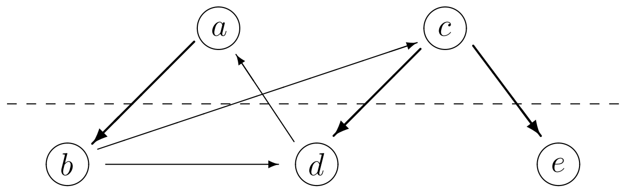

Fig. 2. $\{a, c\}$: independent dominant choice

Reference example *The dominant choice $\{a, c\}$ for instance is independent (see figure 2).*

5.4 Dominant and absorbent kernels

Independence and domination (absorbancy) are tightly related. A conjointly dominant (absorbent) and independent choice is called a dominant (absorbent) kernel (Ber70), (BPR05).

Property 5 *(Ber58) Let $\widetilde{G}(X, \widetilde{\mathrm{R}})$ be a digraph. A dominant (absorbent) kernel Y in $\widetilde{G}$ is a minimal dominant (absorbent) and maximal independent choice.*

Proof: Let us suppose that Y is indeed not a minimal dominant (absorbent) choice. This implies that there exists a dominant (absorbent) choice $Y' \subset Y$ such that Y' is still dominant (absorbent). This implies that $\forall y \in Y \setminus Y'$ there must exist some $y' \in Y'$ such that $\widetilde{\mathrm{R}}(y, y') > 0$ ($\widetilde{\mathrm{R}}(y', y) > 0$). This contradicts the independent quality of the kernel Y.

Let us now suppose that Y is not a maximal independent choice. This implies that there must exist a $Y' \supset Y$ such that Y' is still independent. As Y is a kernel by hypothesis, it is a dominant (absorbent) choice, i.e. $\forall y' \in Y' \backslash Y$ there must exist some $y \in Y$ such that $\widetilde{\mathrm{R}}(y, y') > 0$ ($\widetilde{\mathrm{R}}(y', y) > 0$). This contradicts the independent quality of Y'. $\square$

It is important to see here that a dominant (absorbent) kernel is a minimal dominant (absorbent) choice.

Furthermore one should notice that the set of minimal dominant (absorbent) choices may be much larger than the set of dominant (absorbent) and independent choices in a given digraph. In fact, as seen in Property 2, every finite digraph admits at least one minimal dominant (absorbent) choice. Unfortunately, the existence of an independent and minimal dominant (absorbent) choice is not guaranteed.

Reference example *We can see on figure 1 that the graph admits no independent and absorbent choice.*

Once again, Property 5 gives us two complementary algorithms for the determination of dominant (absorbent) kernels: a first algorithm starting with the choice $Y = X$ and a second algorithm starting with all the singletons.

A choice or relation in a digraph $\widetilde{G}(X, \widetilde{\mathrm{R}})$ is said to have an $\mathcal{L}$*-property* if it has this certain caracteristic in its associated strict median cut crisp digraph $G(X, R)$.

In particular, the relation $\widetilde{\mathrm{R}}$ is $\mathcal{L}$-transitive if the corresponding relation R is transitive in G.

Let us have a look at this particular case, where the outranking relation R is transitive. This is for example the case in a partial order or a weak order.

Property 6 *Let $\widetilde{G}(X, \widetilde{\mathrm{R}})$ be a digraph. If $\widetilde{\mathrm{R}}$ is $\mathcal{L}$-transitive, a minimal dominant (absorbent) choice is also independent.*

Proof: As $\mathcal{L}$-transitivity is defined in the strict median cut crisp digraph, let us verify this property in $G(X, R)$. Let us suppose that a minimal dominant (absorbent) choice Y is not independent. This implies that there exists at least one couple of nodes (y, y') in $Y \times Y$ such that $y \,\mathrm{R}\, y'$ ($y' \,\mathrm{R}\, y$). As Y is a dominant (absorbent) choice, we know by property 1 that every node $y \in Y$ has a non empty dominated (absorbed) private neighbourhood. In particular, $\Gamma^+_Y(y') \neq \emptyset$ ($\Gamma^-_Y(y') \neq \emptyset$). By transitivity, $\Gamma^+_Y(y) \neq \emptyset$ ($\Gamma^-_Y(y) \neq \emptyset$). Therefore,

$Y \setminus \{y'\}$ is also a dominant (absorbent) choice. This is hower in contradiction with Y's minimality for the dominance (absorbence). $\Box$

This gives us an indication on a solution to the choice problem. If the outranking relation is transitive, a best choice recommendation can be given by the concept of kernel. The kernel clearly verifies both principles $\mathcal{R}_\mathbf{1}$ and $\mathcal{R}_\mathbf{2}$. Nevertheless, as already stated, in general the outranking relation is not transitive. In the sequel we present a further argument which shows that the kernel cannot be a solution to the general choice problem.

A digraph $\widetilde{G}(X, \widetilde{\mathrm{R}})$ is said to contain a $\widetilde{L}$-circuit if its associated strict median cut crisp graph G contains a circuit.

Not all digraphs have a dominant (absorbent) kernel. The following property shows the reason for this absence.

Property 7 *If a digraph $\widetilde{G}(X, \widetilde{\mathrm{R}})$ has no dominant (absorbent) kernel, it contains an intransitive $\widetilde{\mathrm{L}}$-circuit of odd order.*

Proof: This property represents the contraposition of Richardson's general result: If a graph contains no circuit of odd order then it has a dominant (absorbent) kernel (see Ber70). $\Box$

This property leads us to think that the concept of kernel is too restrictive for the choice problem. In the following Section we define in a very natural way a more general structure which will help us to solve the choice problem.

5.5 Stable choices and hyper-kernels

A dominant (absorbent) choice Y in a digraph $\widetilde{G}(X, \widetilde{\mathrm{R}})$ is said to be *$P.\alpha$-stable* if and only if the induced subgraph $\widetilde{G}_Y(Y, \widetilde{\mathrm{R}}|_Y)$ does not admit any independent dominant (absorbent) sub-choice $Y' \subset Y$. This means in particular that an acceptable candidate for the choice problem should be $P.\alpha$-stable.

Proposition 1 *Let $\widetilde{G}(X, \widetilde{\mathrm{R}})$ be a digraph. A stable minimal dominant (absorbent) choice Y in $\widetilde{G}$ is either independent or induces a sub-graph $\widetilde{G}_Y(Y, \widetilde{\mathrm{R}}|_Y)$ which contains some $\mathcal{L}$-intransitive $\mathcal{L}$-circuits of odd order.*

Proof: Every $\mathcal{L}$-transitive and/or $\mathcal{L}$-acyclic digraph admits a non empty set of minimal dominant and absorbent choices that are, according to Property 6, necessarily all independent. So let us suppose that $\widetilde{G}$ is not a transitive or acyclic digraph. The observed dominant (absorbent) and independent choices are naturally stable choices. Let us suppose that besides these, the graph $\widetilde{G}$ contains a set of non independent minimal dominant (absorbent) choices. Each of these choices taken individually induce a non-empty subgraph. These subgraphs don't have any dominant (absorbant) kernels because they are minimally dominant (absorbant). According to Property 7, they contain at least one intransitive $\mathcal{L}$-circuit of odd order. □

Reference example *We can observe that the choice $\{a, b, d, e\}$ is simultanously a minimal stable dominant and a minimal stable absorbent choice (see figures 3 and 4).*

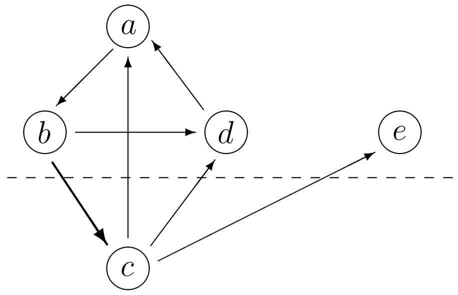

Fig. 3. A minimal stable dominant choice: $\{a, b, d, e\}$

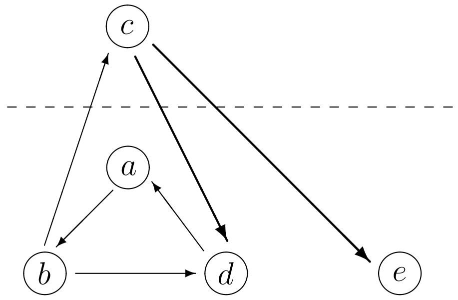

Fig. 4. A minimal stable absorbent choice: $\{a, b, d, e\}$

Proposition 1 naturally leads us to define the concept of *hyper-independence.* A choice Y in $\widetilde{G}$ is said to be *hyper-independent* if it consists of disjoint $\mathcal{L}$-intransitive $\mathcal{L}$-circuits C_p of odd order $(p = 1, 3, \ldots)$ [5]. Consequently a dominant (absorbent) *hyper-kernel* is a hyper-independent dominant (absorbent) choice. Figures 3 and 4 show a dominant and an absorbent hyper-kernel consisting of a C_3 ($\{a, b, d\}$) and a C_1 ($\{e\}$).

Property 8 *Let $\tilde{G(X, \widetilde{\mathrm{R}})}$ be a digraph. A dominant (absorbent) hyper-kernel Y is maximal hyper-independent.*

[5] Singletons are considered as circuits of order 1.

Proof: In case $\widetilde{G}$ has no circuits of odd order $p > 1$, Property 5 applies. In general, let us suppose that a dominant (absorbent) hyper-kernel Y is hyper-independent, but not maximal for this quality. This implies that there must exist a circuit C_p of odd order p in $X \subseteq Y$ such that $Y \cup C_p$ is a again hyper-independent. The odd circuit C_p was thus not included in the dominated (absorbed) neighbourhood of Y. This contradicts the assumption that Y is dominant (absorbent). $\square$

Proposition 2 *Let $\widetilde{G}(X, \widetilde{\mathrm{R}})$ be a digraph. A dominant (absorbent) hyper-kernel is a stable minimal dominant (absorbent) choice Y.*

Proof: Following directly from the definition of a stable minimal dominant (resp. absorbent) choice. $\square$

The following section deals with an algebraic approach to the determination of kernels in a digraph.

6 Algebraic approach to the determination of kernels in digraphs

In this section we recall results from (BPR05).

6.1 The kernel equations

A choice Y in a digraph $\widetilde{G}$ can be characterised by a membership function for each alternative of X. Formally, a characteristic vector $\widetilde{Y}(\cdot)$ of a choice Y is an application from X to $\mathcal{L}$, i.e. a row vector $[\widetilde{Y}(x), x \in X]$. For any x in X, $\widetilde{Y}(x)$ can be interpreted as the degree of credibility of the assertion "*alternative x belongs to the choice Y*".

We now recall the kernel equations which will help us to determine the kernels of the digraph $\widetilde{G}$.

$$(\widetilde{Y} \circ \widetilde{\mathrm{R}})(x) = \max_{y \neq x}[\min(\widetilde{Y}(y), \widetilde{\mathrm{R}}(y,x))] = -\widetilde{Y}(x) \tag{1}$$

$$(\widetilde{\mathrm{R}} \circ \widetilde{Y}^t) = \max_{y \neq x}[\min(\widetilde{\mathrm{R}}(x,y), \widetilde{Y}(y))] = -\widetilde{Y}(x) \tag{2}$$

where $\widetilde{Y}^t(\cdot)$ is the transposed characteristic vector. Let $\mathcal{Y}^{\mathrm{dom}}$ ($\mathcal{Y}^{\mathrm{abs}}$) be the set of solutions of the dominant kernel equations (1)(absorbent kernel equations (2)).

A $\mathcal{L}$-*determined* solution $\widetilde{Y}(\cdot)$ of the kernel equations is a solution of the kernel equations for which $\widetilde{Y}(x) \neq 0$ for each x in X. Let $\mathcal{Y}_0^{\mathrm{dom}}$ ($\mathcal{Y}_0^{\mathrm{abs}}$) be the set of $\mathcal{L}$-determined solutions of the dominant kernel equations (absorbent kernel equations).

Let $\widetilde{Y}_1(\cdot)$ and $\widetilde{Y}_2(\cdot)$ be two elements of $\mathcal{Y}_0^{\mathrm{dom}}$ (or $\mathcal{Y}_0^{\mathrm{abs}}$). $\widetilde{Y}_1(\cdot)$ is said to be *at least as sharp as* $\widetilde{Y}_2(\cdot)$ ($\widetilde{Y}_2(\cdot) \preceq \widetilde{Y}_1(\cdot)$) if and only if for all x in X either $\widetilde{Y}_1(x) \leq \widetilde{Y}_2(x) < 0$ or $0 < \widetilde{Y}_2(x) < \widetilde{Y}_1(x)$.

An $\mathcal{L}$-*dominant* ($\mathcal{L}$-*absorbent*) kernel is a vector $\widetilde{Y}(\cdot)$ which is an $\mathcal{L}$-determined solution of the dominant (absorbent) kernel equation system and which is maximal with respect to the sharpness relation $\preceq$. Let $\mathcal{F}^{\mathrm{dom}}(\widetilde{G})$ ($\mathcal{F}^{\mathrm{abs}}(\widetilde{G})$) be the possibly empty set of $\mathcal{L}$-dominant ($\mathcal{L}$-absorbent) kernels of $\widetilde{G}$.

To each element $\widetilde{Y}(\cdot)$ of $\mathcal{F}^{\mathrm{dom}}(\widetilde{G})$ ($\mathcal{F}^{\mathrm{dom}}(\widetilde{G})$) one can associate its strict median cut crisp choice Y. This choice has also a binary characteristic vector $Y(\cdot)$ for which $Y(x) = 1$ if and only if $\widetilde{Y}(x) > 0$. Let $\mathcal{F}_{>0}^{\mathrm{dom}}(\widetilde{G})$ ($\mathcal{F}_{>0}^{\mathrm{abs}}(\widetilde{G})$)be the set of these crisp choices.

It has been shown in (BPR05) that there exists an isomorphism between the solutions $\mathcal{F}_{>0}^{\mathrm{dom}}(\widetilde{G})$ ($\mathcal{F}_{>0}^{\mathrm{abs}}(\widetilde{G})$) and the dominant (absorbent) kernels of the strict median cut crisp graph G linked to $\widetilde{G}$.

This result is summarised in figure 5.

$$\begin{tikzcd}
\widetilde{G} \arrow[r] \arrow[d] & G \arrow[d] \\
\mathcal{F}^{\mathrm{dom}}(\widetilde{G}) \arrow[r] & \mathcal{F}_{>0}^{\mathrm{dom}}(\widetilde{G})
\end{tikzcd}$$

Fig. 5. Kernels in $\widetilde{G}$ and G

Nevertheless, one should not overestimate this result. In case the outranking relation $\widetilde{\mathrm{R}}$ associated to $\widetilde{G}$ contains 0 values, there may be non-determined solutions to the kernel equations. In that case, this theorem does not apply. This points out once more the particular position of the median value 0 in $\mathcal{L}$. It is a value of undetermination.

In the following Section, we return to the main subject of this paper, namely the choice problem (or $P.\alpha$ problem).

7 Solving the $P.\alpha$ problem by using hyper-kernels

The concept of hyper-kernel has been introduced in Section 5. We see in this Section how to determine the dominating (absorbent) hyper-kernels of a graph $\widetilde{G}(X, \widetilde{\mathrm{R}})$.

8 Conclusion

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