On unlabelled kernels of a digraph

Some empiric results

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Graphs and digraphs

Definition

- A digraph $G(X, R)$ consists of a vertex set X and a binary relation R defined on X.
- A digraph $G(X, R)$ where R is irreflexive and symmetric is also called a (simple) graph.
- A complete graph with n vertices is noted K_n. We denote C_n the symmetric cycle on n vertices.
- A circulant graph, denoted $Circ(\mathbb{Z}_n\{r, s, \ldots\}$, is a digraph with n vertices enumerated $0, 1, \ldots, n-1$ such that each vertex i is linked to vertex $i + r \bmod n$, $i + s \bmod n$ etc.

Special subgraphs

Definition

- A subdigraph of a digraph $G(X, R)$ is a digraph $G'(Y, S)$ such that $Y \subseteq X$ and $S \subseteq R$. If $X = Y$, G' is called a spanning subdigraph.
- A subdigraph is called an induced subdigraph if $y\,S\,y'$ in G' if and only if yRy' in G.
- A clique is an induced subdigraph that is complete. An independent set is an induced empty subdigraph.

Regular digraphs

Definition

- The number of predecessors (resp. successors) of a vertex x in R is called its in-degree, resp. its out-degree.
- The number of neighbors of a vertex x in a graph, is called its degree.
- A k-regular graph is a graph such that each vertex has degree k.
- C_n has degree 2, K_n has degree $n-1$. A 3-regular graph is called cubic.

Isomorphic digraphs

Definition

- Two digraphs are equal if they have the same vertex set X and support the same relation R.
- Two graphs $G(X,R)$ and $G'(Y,S)$ are isomorphic if there is a bijection $\Phi : X \to Y$ such that $x \, R \, x'$ in G if and only if $\Phi(x) \, S \, \Phi(x')$ in G'.

Automorphism group

Definition

- An isomorphism from a digraph G to itself is called an automorphism of G.
- The set of automorphisms of a digraph G is a group called the automorphism group of G and denoted $Aut(G)$.

Comment

- *It is in general a nontrivial task to decide whether two digraphs are isomorphic or that a digraph has no non-identity automorphism.*

Automorphism group (continue)

Definition

- The distance $d_G(x,y)$ between two vertices in a digraph G is the length of the shortest path from x to y.
- The complement $\overline{G}$ of a digraph $G(X,R)$ has the same vertex set X associated with the complement relation $\overline{R}$.

Proposition

- *If x and y are two vertices of G and $g \in Aut(G)$, then $d_G(x,y) = d_{G^g}(x^g, y^g)$.*
- *The automorphism group of a digraph is equal to the automorphism group of its complement.*

Isomorphic subdigraphs

Proposition

- *$Aut(G)$ is a subgroup of the permutation group $Sym(X)$ of the vertex set X. In fact $Aut(K_n)$ only permutes vertices with identical in- and out-degrees.*
- *If $G'(Y,S)$ is a subdigraph of G and $g \in Aut(G)$, then $G'^g(Y^g, S^g)$, such that $Y^g = \{y^g : y \in Y\}$ and $S^g = \{(x^g, y^g) : xSy\}$, is isomorphic to $G'(Y,S)$.*

Kernels in digraphs

Definition

Let $G(X,R)$ be a digraph.

- A dominant (absorbent) choice in G is a non empty subset Y of X such that for all $x \in X - Y$, $\exists y \in Y : y \, R \, x \; (x \, R \, y)$.
- A choice Y in G is independent if for all $x, y \in Y$, $x \not R \, y$.
- A dominant (dominated) and independent choice Y in G is called a dominant (absorbent) kernel of G.

Comment

In symmetric digraphs the kernel concept corresponds to maximal independent sets (Berge 1958).

The kernel equation system

Definition

Let $G(X,R)$ be a digraph and Y a choice function on X. If we denote $\overline{Y}$ the complement choice function :

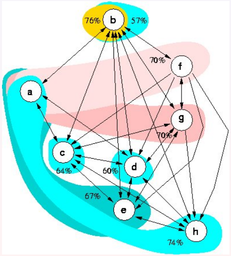
$$\boxed{Y \circ R = \overline{Y} \; (1), \quad R \circ Y = \overline{Y} \; (2)}$$

represent the dominant, resp. absorbent, kernel characteristic systems.

Proposition (Berge 1958)

The solutions to these equation systems (1) and (2) deliver all dominant and absorbent kernels in a digraph.

Kernels in a valued outranking relation



Perny's car selection problem (1992)

Complexity of the kernel extraction

Comment

- *The number k of kernels in a digraph may get very large, even huge. For a digraph of order $n > 20$, k is bounded by $3^{n/3}$ (Tomescu 1990).*
- *Disjoint unions of K_3 give the graphs with the maximum number of kernels !*
- *But high kernel multiplicity arrives also with connected, but highly symmetric graphs.*
- *If C_{20} admits 277 kernels, C_{30} already admits 4610 kernels !*
- *But many of these solutions are in fact isomorphic !*

Kernel orbits and unlabelled kernels

Definition

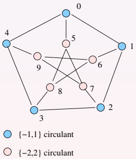
Let $\mathcal{K}$ be the set of all kernels (dominant and absorbent) observed in a digraph G.

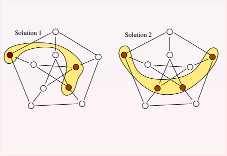
- If Y is a dominant or absorbent kernel of G and $g \in Aut(G)$, then Y^g is a corresponding kernel in G^g. Y and Y^g are called isomorphic kernels.
- The subsets of mutually isomorphic kernels in $\mathcal{K}$ are called kernel orbits.

Proposition

- *Each kernel belongs to a unique orbit, i.e. the set of isomorphic copies of itself.*
- *The orbits give a partition of $\mathcal{K}$, where the individual components are called unlabelled kernels.*

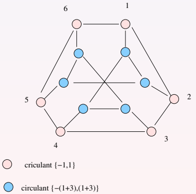
Petersen graph

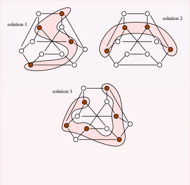




15 kernels, 2 orbits : Solution 1 (10), Solution 2 (5).

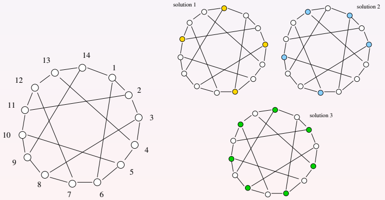
Franklin graph





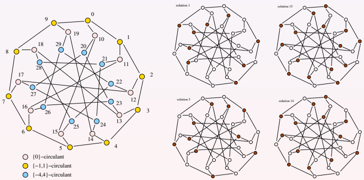
17 kernels, 3 orbits : Solution 1 (12), Solution 2 (3), Solution 3 (2).

Heawood graph



37 kernels, 3 orbits : Solution 1 (21), Solution 2 (14), Solution 3 (2).

Tutte-Coxeter (Levi) graph



3264 kernels, 15 orbits, a.o. Solution 1, card. 9 (90), Solution 5 card. 10 (360), Solution 14 card. 13 (30), Solution 15 card. 15 (2).

St-Nicolas Graphs

Definition

We call St-Nicolas Graph a digraph of order n supporting a unique kernel orbit of size n.

Proposition

- *All K_n are St-Nicolas graphs.*
- *C_5 is also a St-Nicolas graph.*

Comment

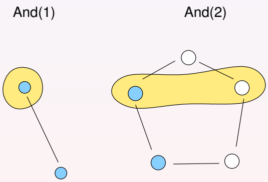
What properties characterise the St-Nicolas graphs ?

Andrásfai graphs

Definition

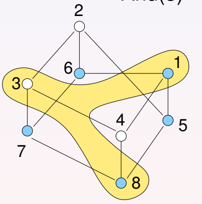
- Let C be the subset of $\mathbb{N}_{3k-1}$ elements congruent to 1 modulo 3.
- The circulant graphs $Circ(\mathbb{Z}_{3k-1}, C)$ for $k \geq 1$ are named after Bela Andrásfai.

Andrásfai graphs



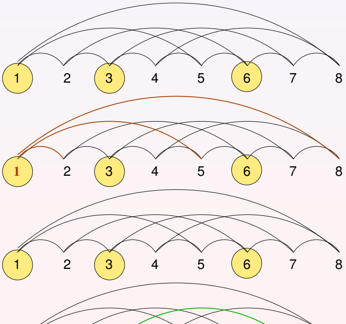
And(1) is K_2, 1 orbit (card 1, 2)
And(2) is C_5, 1 orbit (card 2, 5)

And(3)



And(3) is a Möbius 4-ladder, 1 orbit (card 3,8)

Properties of Andrásfai graphs



Kernel properties of the Andrásfai graphs

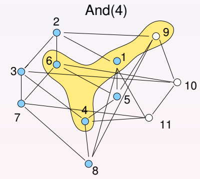
Let $And(k)$ be an Andrásfai graph of order k.

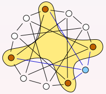
Proposition

1. *$And(k)$ admits k kernels.*
2. *There exists a bijective correspondence between kernels and vertices.*
3. *Each kernel admits a central symmetry axis which devides by two the order of $Aut(And(k)) = 2k$.*

Corollary

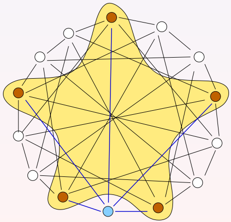
The Andrásfai graphs $And(k)$ are St-Nicolas graphs for $k \geq 1$.





And(4) is $Circ(\mathbb{Z}_{11}, \{1, 4, 7, 10\})$, 1 orbit (card 4,11)

And(5)



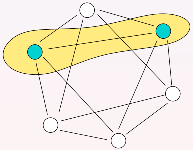
And(5) is $Circ(\mathbb{Z}_{14}, \{1, 4, 7, 10, 13\})$, 1 orbit (card 5,14)

Characterising St-Nicolas graphs

- Properties that don't or may not necessarily characterise the St-Nicolas class of graphs :
  1. Having a specific diameter,
  2. Being reduced or not,
  3. Being triangle-free,
  4. Being a Cayley graph ?
- Properties that necessarily characterise this class :
  1. The fact of being a single circulant graph,
  2. There exists a bijective correspondence between kernels and individual vertices,
  3. The kernels admit a unique central symmetry axis.

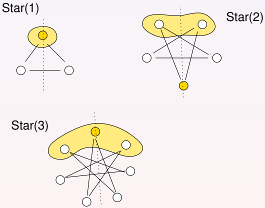
Octahedral graph

Octahedron



The octahedral graph, the circulant {2,-2} on 6 vertices, has 1 orbit, (card 2, 3)

Circulant star graphs



Star(1) is K_3, 1 orbit (St-Nicolas, card 1, 3)
Star(2) is $And(2)$, 1 orbit (St-Nicolas, card 2, 5)
Star(3) is C_7, 1 orbit (St-Nicolas, card 3, 7)

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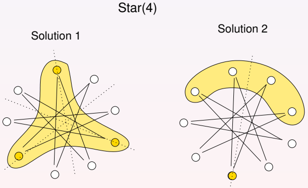
Circulant star graphs (continue)



But, Star(4), $Circ(\mathbb{Z}_9\{4\})$ has 2 orbits :
Kernel 1 (card 3, 3), Kernel 2 (St-Nicolas, card 4, 9)

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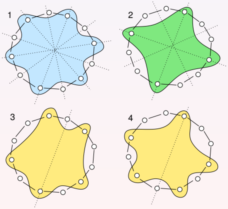
The cycle on 12 vertices



C_{12} has 4 kernel orbits :

1 card 6, 2
2 card 4, 3
3 St-Nicolas card 5
4 St-Nicolas card 5

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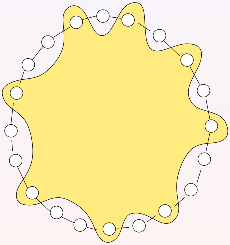
The cycle on 20 vertices



C_{20} has 14 kernel orbits :

- 2 $\times$ 40 card 8
- 9 $\times$ St. Nicolas card 7-9
- 1 $\times$ 10 card 8
- 1 $\times$ 5 card 8
- 1 $\times$ 2 card 10

C_{30} has already 104 kernel orbits.

Cycles of prime order

Definition

We denote $\mathcal{S}_s^k$ (with $s \geq 0$ and $k \geq 1$), k kernel orbits based on s central symmetry axes. A St-Nicolas digraph is $\mathcal{S}_1^1$.

Proposition

C_k *for* $k = 3, 5, 7$ *is* $\mathcal{S}_1^1$ *(St-Nicolas)*
C_{11} *is* $\mathcal{S}_1^2$ *(St-Nicolas)*[2]
C_{13} *is* $\mathcal{S}_1^4$ *(St-Nicolas)*[4]
C_{17} *is* $\mathcal{S}_0^1 + \mathcal{S}_1^5$
C_{19} *is* $\mathcal{S}_0^2 + \mathcal{S}_1^7$
C_{23} *is* $\mathcal{S}_0^8 + \mathcal{S}_1^{12}$

Kernel orbits of circulant graphs

Proposition

- C_9 *is* $\mathcal{S}_1^1 + \mathcal{S}_3^1$
- $Circ(\mathbb{Z}_9, \{1, 3\})$ *is* $\mathcal{S}_1^1$ *(St-Nicolas)*
- C_{10} *is* $\mathcal{S}_1^1 + \mathcal{S}_2^1 + \mathcal{S}_5^1$
- $Circ(\mathbb{Z}_{10}, \{1, 2, 5\})$ *is* $\mathcal{S}_1^1$ *(St-Nicolas)*
- C_{12} *is* $\mathcal{S}_1^2 + \mathcal{S}_4^1 + \mathcal{S}_6^1$
- $Circ(\mathbb{Z}_{12}, \{1, 4, 6\})$ *is* $\mathcal{S}_1^1$ *(St-Nicolas)*
- C_{20} *is* $\mathcal{S}_0^2 + \mathcal{S}_1^9 + \mathcal{S}_2^1 + \mathcal{S}_4^1 + \mathcal{S}_{10}^1$
- $Circ(\mathbb{Z}_{20}, \{1, 2, 4, 10\})$ *is* $\mathcal{S}_1^3$ *(St-Nicolas)*[3]

The St-Nicolas conjecture

Conjecture

- *All cycles* C_n *of prime order* $n \geq 3$ *admit unlabelled kernels only of class* $\mathcal{S}_0^p + \mathcal{S}_1^q$, *with* $p \geq 0$ *and* $q \geq 1$ *;*
- *The circulant graph* $Circ(\mathbb{Z}_n, \{r, s, \ldots\})$ *corresponding to a cycle* C_n *supporting a positive number of kernel orbits with* r, s, *. . ., symmetry axes, admit unlabelled kernels only of class* $\mathcal{S}_0^p + \mathcal{S}_1^q$, *with* $p \geq 0$ *and* $q \geq 1$.

Concluding remarks

- Kernel orbits and unlabelled kernels
- The symmetry class of unlabelled kernels
- The St-Nicolas graphs
- Characterising the symmetry in circulant graphs
- The St-Nicolas conjecture